ON THE POSITIVITY OF KIRILLOV'S CHARACTER FORMULA

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Abstract. We give a direct proof for the positivity of Kirillov's character on the convolution algebra of smooth, compactly supported functions on a connected, simply connected nilpotent Lie group G. Then we use this positivity result to construct a representation of $G \times G$ and establish a $G \times G$-equivariant isometric isomorphism between our representation and the Hilbert–Schmidt operators on the underlying representation of G. Moreover, we show that the same methods apply to coadjoint orbits of other groups such as $\mathrm{SL}(2,\mathbb{R})$ under additional hypotheses that are automatically satisfied in the nilpotent case.

Kirillov's character formula, Coadjoint orbit, Nilpotent Lie group, Positivity, Quantization, Polarization, The GNS construction.

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1. Introduction

In his fundamental paper [12], Kirillov proved that coadjoint orbits of a connected, simply connected nilpotent Lie group correspond, under quantization, to the equivalence classes of its irreducible unitary representations. The theory of geometric quantization due to Kirillov, Kostant [16], and Souriau [22] has shown that this close connection extends to many other groups.

Kirillov's character formula says that the characters of irreducible unitary representations of a Lie group G "should" be given by an equation of the form

$$\operatorname{Tr} \pi_{\mathcal{O}}(\exp X) = j^{-1/2}(X) \int_{\mathcal{O}} e^{i\ell(X)+\sigma} \tag{1.1}$$

where $\mathcal{O}$ is the coadjoint orbit in $\mathfrak{g}^*$ corresponding to $\pi_{\mathcal{O}} \in \widehat{G}$, σ is a canonical symplectic measure on $\mathcal{O}$, and j is the analytic function on $\mathfrak{g}$ defined by the formula

$$j(X) = \det\left(\frac{\sinh(\operatorname{ad} X/2)}{\operatorname{ad} X/2}\right).$$

Kirillov proved his character formula for simply connected nilpotent and simply connected compact Lie groups [12, 13] and conjectured its universality. The validity of this conjecture has been verified for some other classes of Lie groups, most notably for the case of tempered representations of reductive Lie groups by Rossmann [20]. Moreover, Atiyah–Bott [2] and Berline–Vergne [3] following the work of Duistermaat–Heckman [6], have shown that for compact Lie groups, Kirillov's character formula is equivalent to the Weyl character formula.

Our main contributions in this paper are as follows. Rather than construct representations, compute their characters, and compare the results with Kirillov's character formula, we work directly with Kirillov's formula and give direct arguments to prove positivity in the sense of operator algebras. More precisely, we give a direct proof for the positivity of Kirillov's character on the convolution algebra of the Schwartz functions on a connected nilpotent Lie group (Theorem 3.8). The fact that Kirillov's character defines a positive trace is remarkable in light of the Gelfand–Naimark–Segal (GNS) construction from operator algebra theory, since GNS implies, roughly speaking, that any positive linear functional on a Lie group is the character of a group representation. In the last section, we use our positivity result and the GNS to construct a representation of $G \times G$ for a connected, simply connected nilpotent Lie group G. Then in Theorem 5.3 we discuss the relation between the representations that we have obtained from the GNS construction, and the Kirillov representations corresponding to the coadjoint orbits of G. Indeed, we prove that there is a $G \times G$-equivariant isometric isomorphism between our representation and the Hilbert–Schmidt operators on the underlying representation of G. We deal with nilpotent groups this way, thereby recovering some of the first results obtained by Kirillov. As seen in Theorem 4.9, the same methods apply to coadjoint orbits of other groups such as $\mathrm{SL}(2, \mathbb{R})$ in the presence of supplementary geometric hypotheses on the orbits that are automatically satisfied in the nilpotent case. One hypothesis is the existence of a real polarization; a second is a completeness hypothesis on the (locally affine) fibers of the associated Lagrangian fibration.

2. Some Notations and Background Material

In this section we fix some notations and collect several standard results that we will use freely in the sequel.

Definition 2.1. Let G be a locally compact group G with a left Haar measure $d\mu$. The *modular function* $\Delta\colon G \to (0, \infty)$ is defined via

$$\Delta(y) \int_G f(xy)\, d\mu(x) = \int_G f(x)\, d\mu(x)$$

for x and y in G and all $f \in L^1(G)$ so that

$$\int_G f(x^{-1})\, d\mu(x) = \int_G f(x)\Delta(x^{-1})\, d\mu(x).$$

If G is a Lie group, then it is well known that $\Delta(g) = |\det \mathrm{Ad}(g^{-1})|$. This implies, for instance, that any connected nilpotent Lie group is unimodular, that is, for such groups the modular function is constant and everywhere equal to one.

Definition 2.2. Let G be a locally compact group with a left Haar measure $d\mu$. Given functions f and g on G, their *convolution* $f * g$ is the function on G defined by

$$\begin{aligned} f * g(x) &= \int_G f(y) g(y^{-1}x)\, d\mu(y) \\ &= \int_G f(xy) g(y^{-1})\, d\mu(y) \end{aligned}$$

whenever one (and hence both) of these integrals makes sense.

Let (π, V_π) be a unitary representation of a locally compact group G. Then π induces a continuous homomorphism of Banach $*$-algebras from $L^1(G)$ to the space of bounded operators $\mathcal{B}(V_\pi)$ via Bochner integration. By abuse of notation, we denote this induced $*$-homomorphism by π. Hence,

$$\pi\colon L^1(G) \to \mathcal{B}(V_\pi) \quad \text{and} \quad \pi(f) = \int_G f(g)\pi(g)\,dg.$$

Here we are using the facts that $L^1(G)$ is a Banach algebra under convolution, which is just the usual group ring $\mathbb{C}G$ if G is finite, and that $L^1(G)$ has a natural isometric (conjugate-linear) involution $f \mapsto f^*$, where

$$f^*(x) = \overline{f(x^{-1})}\Delta(x^{-1}).$$

Note that these definitions are formulated so that $f * g = L(f)(g)$ for the left regular representation $(L, L^1(G))$.

Definition 2.3. We say that an irreducible unitary representation π of a Lie group G has a *global or Harish-Chandra character* Θ if $\pi(f)$ is of trace class for all $f \in C_c^\infty(G)$ and moreover $f \mapsto \Theta(f) = \operatorname{Tr}\pi(f)$ is a distribution.

The next two results provide sufficient conditions for $\pi(f)$ to be trace class.

Theorem 2.4 (Harish-Chandra [8])**.** *Let G be a connected semisimple Lie group with finite center. Then for every irreducible unitary representation π of G and every $f \in C_c^\infty(G)$, the operator $\pi(f)$ is trace class and the map $f \mapsto \operatorname{Tr}\pi(f)$ is a distribution on G.*

Theorem 2.5 (Kirillov [12])**.** *Let G be a nilpotent Lie group. Then for every irreducible unitary representation π of G and every Schwartz function $f \in \mathcal{S}(G)$, the operator $f \mapsto \operatorname{Tr}\pi(f)$ is trace class and the map $f \mapsto \operatorname{Tr}\pi(f)$ is a tempered distribution on G.*

Let (π, V_π) be an irreducible unitary representation. Observe that the above discussions imply that $\pi(f * f^*) = \pi(f)\pi(f)^*$. Thus, the distributional character of (π, V_π), whenever defined, is *convolution-positive* (or *positive* for short) in the sense that

$$\operatorname{Tr}\pi(f * f^*) = \operatorname{Tr}\pi(f)\pi(f)^* = \|\pi(f)\|_{\mathrm{HS}}^2 \geq 0. \tag{2.1}$$

To close this section, we introduce Kirillov's character formula which says that the characters χ of irreducible unitary representations of a Lie group G "should" be given by an equation of the form

$$\chi(\exp X) = j^{-1/2}(X)\int_{\mathcal{O}} e^{i\ell(X)}\,d\mu_{\mathcal{O}}(\ell). \tag{2.2}$$

Here $\mathcal{O}$ is a coadjoint orbit in $\mathfrak{g}^*$, $\mu_{\mathcal{O}}$ is the canonical (or Liouville) symplectic measure on $\mathcal{O}$, and j is the Jacobian of the exponential map $\exp\colon \mathfrak{g} \to G$ given by the formula

$$j(X) = \det\left(\frac{\sinh(\operatorname{ad} X/2)}{\operatorname{ad} X/2}\right)$$

whenever G is unimodular. This character formula should be interpreted as an equation of distributions on a certain space of test functions on $\mathfrak{g}$ as follows. For all smooth functions f compactly supported in a sufficiently small neighborhood of the origin in $\mathfrak{g}$,

$$\operatorname{Tr}\int_{\mathfrak{g}} f(X)\pi(\exp X)\,dX = \int_{\mathcal{O}}\int_{\mathfrak{g}} e^{i\ell(X)}f(X)j^{-1/2}(X)\,dX d\mu_{\mathcal{O}}(\ell) \tag{2.3}$$

where π is the representation with character χ.

Theorem 2.6 (Kirillov [12])**.** *Suppose G is a connected nilpotent Lie group. Then the irreducible unitary representations of G are in natural one-to-one correspondence with the integral orbits of G. Moreover, if G is simply connected and π is an irreducible unitary representation of G with the associated coadjoint orbit $\mathcal{O}$, then* (2.3) *holds.*

3. Nilpotent Lie Groups

Nilpotent Lie groups and their representations have been studied extensively in the literature. From the many papers by Corwin, Greanleaf, Lipsman, Pukanzsky and others we only cite [15], [18], and [21] and refer the reader to [4] for a more comprehensive list of bibliographies. For a nilpotent Lie group G, let $\chi_{\mathcal{O}}$ denote Kirillov's character corresponding to a coadjoint orbit $\mathcal{O}$ of G. That is,

$$\chi_{\mathcal{O}}(f) = \int_{\mathcal{O}} \widehat{f \circ \exp}(\ell)\, d\mu_{\mathcal{O}}(\ell) = \int_{\mathcal{O}} \int_{\mathfrak{g}} f(\exp X) e^{i\ell(X)}\, dX\, d\mu_{\mathcal{O}}(\ell), \quad f \in \mathcal{S}(G).$$

In this section we show directly that if G is a connected nilpotent Lie group, then for all $f \in \mathcal{S}(G)$,

$$\chi_{\mathcal{O}}(f * f^*) \geq 0.$$

We first prove this result for the important case of the Heisenberg group where computations are short and insightful, and then we provide a separate proof for the general case.

The Heisenberg group H can be realized by the 3×3 upper triangular matrices

$$H = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\}$$

with the usual matrix multiplication. The Lie algebra $\mathfrak{h}$ of the Heisenberg group has three generators X, Y, and Z satisfying the commutation relation $[X, Y] = Z$.

Example 3.1 (Positivity of Kirillov's Character for H)**.** Choose $\ell \in \mathfrak{h}^*$ with $\ell(Z) = \gamma \neq 0$ so that the coadjoint orbit $\mathcal{O}$ through ℓ is the plane $Z^* = \gamma$ in the $X^*Y^*Z^*$-coordinate system in $\mathfrak{h}^*$. Let $f \in \mathcal{S}(H)$ be a function in the Schwartz space of the Heisenberg group and identify the coadjoint orbit $\mathcal{O}$ and the Lie algebra $\mathfrak{h}$ with the plane $\mathbb{R}^2 \times \{\gamma\} = \{(\alpha, \beta, \gamma) \mid \alpha, \beta \in \mathbb{R}\}$ and $\mathbb{R}^3 = \{(a, b, c) \mid a, b, c \in \mathbb{R}\}$, respectively. The Liouville measure on $\mathcal{O}$ is $\mu_{\mathcal{O}} = \frac{1}{2\pi\gamma} dX \wedge dY$. We compute

$$\begin{aligned}
\chi_{\mathcal{O}}(f) &= \int_{\mathcal{O}} \int_{\mathfrak{h}} f(\exp X) e^{i\phi(X)}\, dX\, d\mu_{\mathcal{O}}(\phi) \\
&= \frac{1}{2\pi|\gamma|} \int_{\mathcal{O}} \int_{\mathfrak{h}} f(\exp(a, b, c)) e^{i(a\alpha + b\beta + c\gamma)}\, da\, db\, dc\, d\alpha\, d\beta \\
&= \frac{1}{2\pi|\gamma|} \int_{c} \int_{(\alpha,\beta)} \int_{(a,b)} f(\exp(a, b, c)) e^{i(a\alpha + b\beta + c\gamma)}\, da\, db\, d\alpha\, d\beta\, dc \\
&= \frac{2\pi}{|\gamma|} \int_{c} f(\exp(0, 0, c)) e^{ic\gamma}\, dc
\end{aligned}$$

where in the last step we have used the Fourier inversion theorem. Thus, we have found the following simplified form of Kirillov's formula for the coadjoint orbit $\mathcal{O}$

of the Heisenberg group:

$$\chi_{\mathcal{O}}(f) = \frac{2\pi}{|\gamma|} \int_{\mathbb{R}} f(\exp tZ) e^{it\gamma}\, dt, \quad f \in \mathcal{S}(H).$$

Now we use this to check the positivity of $\chi_{\mathcal{O}}$. First note that

$$\chi_{\mathcal{O}}(f * f^*) = \frac{2\pi}{|\gamma|} \int_{\mathbb{R}} \int_H f(\exp tZ \cdot h) \overline{f(h)} e^{it\gamma}\, d\lambda(h)\, dt,$$

where $d\lambda = dx \wedge dy \wedge dz$ is the Haar measure on H, and that we can set $h = \exp(xX + yY + zZ)$ due to the surjectivity of $\exp\colon \mathfrak{h} \to H$. Since $\exp Z$ is in the center $Z(H)$ of H we may rearrange the last expression to find

$$\begin{aligned}
&\frac{2\pi}{|\gamma|} \int_{\mathbb{R}} \int_{\mathbb{R}^3} f(\exp(xX + yY + (z+t)Z)) \overline{f(\exp(xX + yY + zZ))} e^{it\gamma}\, d\lambda\, dt \\
&= \frac{2\pi}{|\gamma|} \int_{\mathbb{R}} \int_{\mathbb{R}^3} f(\exp(xX + yY + (z+t)Z)) e^{i(z+t)\gamma} \overline{f(\exp(xX + yY + zZ)) e^{iz\gamma}}\, d\lambda\, dt \\
&= \frac{2\pi}{|\gamma|} \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}} f(\exp(xX + yY + zZ)) e^{iz\gamma}\, dz \right|^2 dy\, dx \geq 0,
\end{aligned}$$

as desired. Kirillov's character is also positive for any point-orbit $\mathcal{O} = \{\ell\}$ in the X^*Y^*-plane:

$$\begin{aligned}
\chi_{\mathcal{O}}(f * f^*) = \int_{\mathfrak{h}} f * f^*(\exp X) e^{i\ell(X)}\, dX &= \int_H f * f^*(x) e^{i\ell(\log x)}\, d\lambda(x) \\
&= \int_H \int_H f(xy) \overline{f(y)}\, d\lambda(y) e^{i\ell(\log x)}\, d\lambda(x) \\
&= \int_H \int_H f(x) \overline{f(y)} e^{i\ell(\log xy^{-1})}\, d\lambda(x)\, d\lambda(y) \\
&= \left| \int_H f(x) e^{i\ell(\log x)}\, d\lambda(x) \right|^2 \geq 0.
\end{aligned}$$

The last equality is due to the facts that if $\xi, \eta \in \mathfrak{h}$, then

$$\exp \xi \exp \eta = \exp(\xi + \eta + \frac{1}{2}[\xi, \eta])$$

and that since $[\xi, \eta] \in \mathfrak{z}(\mathfrak{h})$, $\ell([\xi, \eta]) = 0$. Therefore, the map from G to $\mathbb{C}$ defined by $x \mapsto \ell(\log x)$ is a group homomorphism.

To proceed with the case of general nilpotent Lie groups, we recall two definitions.

Definition 3.2. Let $\mathfrak{g}$ be a real finite-dimensional Lie algebra and let $\ell \in \mathfrak{g}^*$. A subalgebra $\mathfrak{m} \subset \mathfrak{g}$ is said to be a *real polarization* subordinate to ℓ provided that $\ell([X, \mathfrak{m}]) = 0$ if and only if $X \in \mathfrak{m}$.

If $\mathfrak{g}$ is a nilpotent Lie algebra, and $\ell \in \mathfrak{g}^*$, then there exists a polarizing subalgebra subordinate to ℓ. See [4, Theorem 1.3.3].

Definition 3.3. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then the *stabilizer subgroup* associated with $\ell \in \mathfrak{g}^*$ is

$$R_\ell = \{g \in G \mid \mathrm{Ad}^*(g)\ell = \ell\}$$

which has the corresponding *radical Lie algebra*

$$\mathfrak{r}_\ell = \{X \in \mathfrak{g} \mid \mathrm{ad}^*(X)\ell = 0\}.$$

Throughout, we shall use the notations introduced in these definitions without further comment. Any polarizing subalgebra subordinate to ℓ lies halfway between the radical $\mathfrak{r}_\ell$ and the Lie algebra $\mathfrak{g}$. This motivates the following lemma.

Lemma 3.4. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and a polarizing subalgebra $\mathfrak{m}$ subordinate to an element $\ell \in \mathfrak{g}^*$. Then the mapping*

$$\begin{aligned}\theta\colon M/R_\ell &\to (\mathfrak{g}/\mathfrak{m})^* \\ mR_\ell &\mapsto m \cdot \ell - \ell.\end{aligned}$$

is a diffeomorphism, provided that the orbit of ℓ under M, $\mathrm{Ad}^*(M)\ell$, is closed in $\mathfrak{g}^*$. In particular, θ is automatically a diffeomorphism if G is a connected nilpotent Lie group.

Proof. Note that θ is well defined, because by the identity $\mathrm{Ad}^*(\exp X) = e^{\mathrm{ad}^* X}$ we have $m \cdot \ell(P) = \ell(P)$ for $P \in \mathfrak{m}$ and $m \in M$. Injectivity is clear since $m_1 \cdot \ell - \ell = m_2 \cdot \ell - \ell$ gives $m_1^{-1} m_2 \in R_\ell$ and hence $m_1 R_\ell = m_2 R_\ell$. Surjectivity of θ, on the other hand, is equivalent to surjectivity of its lift $\tilde{\theta}$ to the group M defined by $m \mapsto m \cdot \ell - \ell$. Computing the differential of the equivariant map $\tilde{\theta}$ at a point $g \in M$ one infers that

$$\tilde{\theta}_{*,g}(X) = \frac{d}{dt}\Big|_{t=0} \tilde{\theta}(g \exp tX) = g \cdot ((\mathrm{ad}^* X)\ell) \quad \text{ for } X \in T_g M,$$

and

$$\begin{aligned}\operatorname{rank} \tilde{\theta}_{*,g} &= \dim T_g M - \dim\{X \in T_g M \mid (\mathrm{ad}^* X)\ell = 0\} \\ &= \dim \mathfrak{m} - \dim \mathfrak{r}_\ell.\end{aligned}$$

Consequently, $\tilde{\theta}$ is a submersion, indeed a local diffeomorphism, and hence an open map. Because $\mathrm{Ad}^*(M)\ell$ is closed in $\mathfrak{g}^*$ by assumption, we conclude that the image of $\tilde{\theta}$ is a closed submanifold of $(\mathfrak{g}/\mathfrak{m})^*$ and the surjectivity of θ follows. The last assertion of the theorem is a consequence of a result of Chevalley and Rosenlicht [4, Theorem 3.1.4] which states that if G acts unipotently on a real vector space V, then the G-orbits are closed in V. □

Variations of the closedness assumption used in the statement of this lemma are known as the *Pukanzsky condition* in the literature.

Theorem 3.5 (Weil's formula)**.** *Let G be a locally compact group, and let H be a closed subgroup. There exists a G-invariant Radon measure $\nu \neq 0$ on the quotient G/H if and only if the modular functions Δ_G and Δ_H agree on H. In this case, the measure ν is unique up to a positive scalar. Given Haar measures on G and H, there is a unique choice for ν such that for every $f \in C_c(G)$ one has the quotient integral formula*

$$\int_G f(g)\, dg = \int_{G/H} \int_H f(xh)\, dh\, d\nu(xH).$$

To lighten the notation we shall write dxH for $d\nu(xH)$. We will have an occasion for using a slightly more general form of this theorem involving a tower of three groups which we now state.

Corollary 3.6. *Let G be a locally compact group with closed subgroups H and K such that $K \subset H \subset G$. Then*

$$\Delta_G|_H = \Delta_H \quad \textit{and} \quad \Delta_H|_K = \Delta_K$$

if and only if there exist nonzero suitably normalized invariant Radon measures on the quotient spaces such that the equality

$$\int_{G/K} f(g)\,dgK = \int_{G/H}\int_{H/K} f(xh)\,dhK\,dxH$$

holds for any $f \in C_c(G/K)$.

Proof. First assume the equality of restrictions of modular functions. Let $F \in C_c(G)$ and apply Theorem 3.5 twice to obtain

$$\begin{aligned}\int_G F(t)\,dt &= \int_{G/K}\int_K F(gk)\,dk\,dgK \\ &= \int_{G/H}\int_H F(xs)\,ds\,dxH = \int_{G/H}\left(\int_{H/K}\int_K F(xhk)\,dk\,dhK\right)dxH.\end{aligned}$$

Fix $f \in C_c(G/K)$ and choose a function $\alpha \in C_c(G)$ with the property that for every $gK \in \text{supp}\,(f)$ we have $\int_K \alpha(gk)\,dk = 1$, then substitute $F = f\alpha$. For the standard proof of existence of such α we refer to [7, Lemma 2.47]. The converse is immediate from the first part of Theorem 3.5. $\square$

For a geometric proof of this "chain rule for integration" formula see [9, Proposition 1.13].

The following theorem is proved by Lipsman [17]. Since we are mainly concerned with establishing positivity results, we shall not go into any discussion of the normalizations of measures and instead refer the reader to [11] for the details.

Theorem 3.7. *Let G be a connected nilpotent Lie group, $\mathfrak{m}$ a polarizing subalgebra subordinate to some $\ell \in \mathfrak{g}^*$, and $M = \exp\mathfrak{m}$. Then for any $f \in \mathcal{S}(G)$,*

$$\chi_{\mathcal{O}}(f) = \int_{G/M}\int_M f(xpx^{-1})e^{i\ell(\log p)}\,dp\,dxM, \tag{3.1}$$

where $\chi_{\mathcal{O}}$ is the Kirillov character for the coadjoint orbit $\mathcal{O}$ through ℓ.

If G is a connected, simply connected nilpotent Lie group, Kirillov's character is known to be positive for any coadjoint orbit of G in view of Theorems 2.5 and 2.6, and Equation (2.1). We are now ready to prove this positivity result directly and without giving any reference to the underlying representation.

Theorem 3.8. *Let G be a connected (but not necessarily simply connected) nilpotent Lie group. Then for any coadjoint orbit $\mathcal{O}$ of G, Kirillov's character $\chi_{\mathcal{O}}$ is positive on the convolution algebra of Schwartz functions on G.*

Proof. Let $f \in \mathcal{S}(G)$. We use Theorem 3.7 to prove that $\chi_{\mathcal{O}}(f * f^*) \geq 0$.

$$\begin{aligned}\chi_{\mathcal{O}}(f * f^*) &= \int_{G/M}\int_M f * f^*(xpx^{-1})e^{i\ell(\log p)}\,dp\,dxM \\ &= \int_{G/M}\left(\int_M\int_G f(xpx^{-1}g)f^*(g^{-1})e^{i\ell(\log p)}\,dg\,dp\right)dxM \\ &= \int_{G/M}\left(\int_M\int_G f(xpx^{-1}g)\overline{f(g)}e^{i\ell(\log p)}\,dg\,dp\right)dxM.\end{aligned}$$

Applying the change of variable $g \mapsto xg^{-1}$ to the innermost integral and using the unimodularity of G the last integral simplifies to

$$\int_{G/M} \left(\int_M \int_G f(xpg^{-1})\overline{f(xg^{-1})} e^{i\ell(\log p)} \, dg \, dp \right) dxM.$$

Next, we apply the quotient integral formula to G to decompose the measure over M and G/M

$$\int_{G/M} \left(\int_M \int_{G/M} \int_M f(xpq^{-1}y^{-1})\overline{f(xq^{-1}y^{-1})} e^{i\ell(\log p)} \, dq \, dyM \, dp \right) dxM.$$

Finally, we change the order of the two integrations in the middle, use the change of variable $p \mapsto p^{-1}q$ and unimodularity of M to find

$$\begin{aligned} &\chi_{\mathcal{O}}(f * f^*) \\ &\quad = \int_{G/M} \int_{G/M} \int_M \int_M f(xp^{-1}y^{-1})\overline{f(xq^{-1}y^{-1})} e^{i\ell\left(\log p^{-1}q\right)} \, dq \, dp \, dyM \, dxM \\ &\quad = \int_{G/M} \int_{G/M} \left| \int_M f(xp^{-1}y^{-1}) e^{-i\ell(\log p)} \, dp \right|^2 dyM \, dxM \geq 0. \end{aligned}$$

$\square$

4. SL$(2, \mathbb{R})$ and Beyond

Rossmann has shown that Kirillov's formula is valid for characters of irreducible tempered representations of semisimple Lie groups. (Characters of non-tempered irreducible representations, on the other hand, usually do not arise as Fourier transforms of invariant measures on coadjoint orbits.)

Many coadjoint orbits of non-nilpotent Lie groups fall into the scope of the method used in Section 3 for nilpotent Lie groups. In this section we make some adjustments to our methods for nilpotent Lie groups and explain how they can be applied to prove the positivity of Kirillov's character in a broader context by focusing on the special linear group SL$(2, \mathbb{R})$. To streamline the notation, we write

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \text{ and } Y = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{4.1}$$

for a basis of the three dimensional Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ of 2×2 traceless matrices.

To carry out our new computations, we need to be able to decompose the Haar measure of a locally compact group G over a closed subgroup H and the quotient G/H when G/H does not necessarily admit a G-invariant measure.

Definition 4.1. Let G be a locally compact group and H a closed subgroup of G. A Radon measure μ on G/H is called *quasi-invariant* under G if there exist functions λ_g defined on G/H such that for all $g \in G$ and $f \in C_c(G/H)$

$$\int_{G/H} f(\Lambda_g x) \, d\mu(xH) = \int_{G/H} f(x)\lambda_g(x) \, d\mu(xH),$$

where $\Lambda_g(xH) = g^{-1}xH$.

Note that if $\lambda_g = 1$ for all $g \in G$, then μ is invariant under G, and hence quasi-invariance extends the notion of invariance.

The next theorem generalizes Weil's formula, Theorem 3.5, that we used for integration over nilpotent Lie groups.

Theorem 4.2 (Mackey–Bruhat)**.** *Let G be a locally compact group. Given a closed subgroup H of G, there is always a continuous, strictly positive solution ρ of the functional equation*

$$\rho(xh) = \rho(x)\frac{\Delta_H(h)}{\Delta_G(h)}, \quad x \in G, h \in H. \tag{4.2}$$

Moreover, there is a quasi-invariant measure $d_\rho xH$ on G/H such that

$$\int_G f(g)\rho(g)\,dg = \int_{G/H}\int_H f(xh)\,dh\,d_\rho xH.$$

Remark 4.3. Henceforth we will assume that our quotient spaces are equipped with measures as in Theorem 4.2 and we will drop the index ρ in the measure. One can show that in this situation

$$\lambda_g(xH) = \frac{\rho(gx)}{\rho(x)} \quad \text{for} \quad x, g \in G.$$

See Definition 4.1 and, for instance, [19, Proposition 8.1.4]. If G/H carries a G-invariant measure, then we shall assume that $\rho \equiv 1$; this happens, for instance, when we study quotients diffeomorphic to coadjoint orbits which naturally carry invariant symplectic measures.

Example 4.4. In this example we compute the ρ function for two pairs of groups consisting of the Lie group $G = \mathrm{SL}(2, \mathbb{R})$ and its closed subgroups M and R that we introduce below. Consider the basis $\{H, X, Y\}$ for the Lie algebra $\mathfrak{g}$ as in (4.1) and let $\ell = H^*$ be the functional dual to H with respect to this basis. Then a polarizing subalgebra of $\mathfrak{g}$ subordinate to ℓ is given by the upper triangular matrices

$$\mathfrak{m} = \mathrm{Span}\{H, X + Y\}.$$

Let M denote the subgroup generated by $\mathfrak{m}$. A short calculation shows that

$$M = \exp \mathfrak{m} = \left\{ \begin{bmatrix} a & * \\ 0 & a^{-1} \end{bmatrix} \;\middle|\; a > 0 \right\}, \text{ and}$$

$$R = \{g \in G \mid \mathrm{Ad}^*(g)\ell = \ell\} = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \;\middle|\; a \in \mathbb{R}^\times \right\}.$$

To solve functional equation (4.2) for ρ, given the pairs (G, M) and (M, R), we need to know the modular functions of M and R. Note that $\Delta_M(t) = \det \mathrm{Ad}(t^{-1})$ for $t \in M$. More explicitly,

$$\Delta_M \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} = a^{-2}, \quad a > 0.$$

Also, $\Delta_R(t) = \det \mathrm{Ad}(t^{-1})$ for $t \in R$, which implies

$$\Delta_R \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} = 1, \quad a \in \mathbb{R}^\times.$$

Since G is unimodular, $\Delta_G \equiv 1$, and therefore by (4.2),

$$\begin{aligned} \rho_{(G,M)}(gp) &= \rho_{(G,M)}(g)\frac{\Delta_M(p)}{\Delta_G(p)} \\ &= \rho_{(G,M)}(g)a^{-2}, \text{ for } g \in G, \text{ and } p = \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \in M. \end{aligned} \tag{4.3}$$

The function $\rho_{(G,M)}\colon G \to \mathbb{R}$ defined by

$$\rho_{(G,M)} \begin{bmatrix} g_1 & g_2 \\ g_3 & g_4 \end{bmatrix} = \frac{1}{g_1^2 + g_3^2}$$

solves functional equation (4.3). Likewise, since $\Delta_R \equiv 1$ as noted above,

$$\begin{aligned} \rho_{(M,R)}(pr) &= \rho_{(M,R)}(p)\frac{\Delta_R(r)}{\Delta_M(r)} \\ &= \rho_{(M,R)}(p)a^2, \text{ for } p \in M, \text{ and } r = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \in R. \end{aligned} \tag{4.4}$$

The function $\rho_{(M,R)}\colon M \to \mathbb{R}$ defined by $\rho_{(M,R)} = \Delta_M^{-1}$, that is,

$$\rho_{(M,R)} \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} = a^2$$

solves functional equation (4.4).

Now we record a corollary to Theorem 4.2.

Corollary 4.5. *Let G be a locally compact group with closed subgroups H and K such that $K \subset H \subset G$. Then there exist suitably normalized quasi-invariant measures on the quotient spaces such that the equality*

$$\int_{G/K} f(g)\, dgK = \int_{G/H} \int_{H/K} f(xh) \frac{\rho_{(G,K)}(xh)}{\rho_{(G,H)}(xh)\rho_{(H,K)}(h)}\, dhK\, dxH$$

holds for any $f \in C_c(G/K)$.

Proof. Let $F \in C_c(G)$ and apply Theorem 4.2 twice to obtain

$$\begin{aligned} \int_G F(t)\, dt &= \int_{G/K} \int_K \frac{F(gk)}{\rho_{(G,K)}(gk)}\, dk\, dgK \\ &= \int_{G/H} \int_H \frac{F(xs)}{\rho_{(G,H)}(xs)}\, ds\, dxH \\ &= \int_{G/H} \left(\int_{H/K} \int_K \frac{F(xhk)}{\rho_{(G,H)}(xhk)\rho_{(H,K)}(hk)}\, dk\, dhK \right) dxH. \end{aligned}$$

Fix $f \in C_c(G/K)$ and choose a function $\alpha \in C_c(G)$ with the property that for every $gK \in \operatorname{supp}(f)$ we have $\int_K \alpha(gk)\, dk = 1$, then substitute $F = f\alpha\rho_{(G,K)}$. For the standard proof of existence of such α we refer to [7, Lemma 2.47]. The above computation implies

$$\begin{aligned} \int_{G/K} f(g)\, dgK &= \int_{G/H} \int_{H/K} \int_K \frac{f(xhk)\alpha(xhk)\rho_{(G,K)}(xhk)}{\rho_{(G,H)}(xhk)\rho_{(H,K)}(hk)}\, dk\, dhK\, dxH \\ &= \int_{G/H} \int_{H/K} f(xh) \int_K \frac{\alpha(xhk)\rho_{(G,K)}(xh)\frac{\Delta_K(k)}{\Delta_G(k)}}{\rho_{(G,H)}(xh)\frac{\Delta_H(k)}{\Delta_G(k)}\rho_{(H,K)}(h)\frac{\Delta_K(k)}{\Delta_H(k)}}\, dk\, dhK\, dxH \\ &= \int_{G/H} \int_{H/K} f(xh) \frac{\rho_{(G,K)}(xh)}{\rho_{(G,H)}(xh)\rho_{(H,K)}(h)}\, dhK\, dxH. \end{aligned}$$

$\square$

For a Lie group G with Lie algebra $\mathfrak{g}$ write $j_{\mathfrak{g}}$ for the Jacobian of the exponential map $\exp\colon \mathfrak{g} \to G$, with reference to the Lebesgue measure on $\mathfrak{g}$ and the left Haar measure on G. According to a classical result of F. Schur,

$$j_{\mathfrak{g}}(X) = \det\left(\frac{\mathrm{id} - e^{-\operatorname{ad} X}}{\operatorname{ad} X}\right), \quad X \in \mathfrak{g}.$$

In the next example we compute the j function for $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ and one of its subalgebras and point out a relation between the two.

Example 4.6. Consider the polarizing subalgebra

$$\mathfrak{m} = \operatorname{Span}\{X + Y, H\}$$

subordinate to $\ell = H^*$ consisting of upper triangular matrices in $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$. For $W = aH + b(X+Y)$, the matrix of $\operatorname{ad}_{\mathfrak{m}} W\colon \mathfrak{m} \to \mathfrak{m}$ with respect to the basis $\mathfrak{B}' = \{X+Y, H\}$ of $\mathfrak{m}$ is given by

$$[\operatorname{ad}_{\mathfrak{m}} W]_{\mathfrak{B}'} = \begin{bmatrix} 2a & -2b \\ 0 & 0 \end{bmatrix}.$$

Therefore,

$$j_{\mathfrak{m}}(W) = \det\left(\frac{\mathrm{id} - e^{-\operatorname{ad}_{\mathfrak{m}} W}}{\operatorname{ad}_{\mathfrak{m}} W}\right) = \frac{1 - e^{-2a}}{2a} \tag{4.5}$$

by the spectral mapping theorem applied to the eigenvalues of $\operatorname{ad}_{\mathfrak{m}} W$, namely $2a$ and 0. Extend the basis $\mathfrak{B}'$ of $\mathfrak{m}$ to the basis $\mathfrak{B} = \{X+Y, H, X\}$ of $\mathfrak{g}$. Then the matrix of $\operatorname{ad}_{\mathfrak{g}} W\colon \mathfrak{g} \to \mathfrak{g}$ with respect to the basis $\mathfrak{B}$ is

$$[\operatorname{ad}_{\mathfrak{g}} W]_{\mathfrak{B}'} = \begin{bmatrix} 2a & -2b & 2a \\ 0 & 0 & 2b \\ 0 & 0 & -2a \end{bmatrix}.$$

Therefore,

$$j_{\mathfrak{g}}(W) = \det\left(\frac{\mathrm{id} - e^{-\operatorname{ad}_{\mathfrak{g}} W}}{\operatorname{ad}_{\mathfrak{g}} W}\right) = \left(\frac{1 - e^{-2a}}{2a}\right)\left(\frac{1 - e^{2a}}{-2a}\right) = \frac{(e^{a} - e^{-a})^2}{4a^2} \tag{4.6}$$

by the spectral mapping theorem applied to the eigenvalues of $\operatorname{ad}_{\mathfrak{g}} W$, namely $2a$, 0, and $-2a$. Equations (4.5), and (4.6) reveal an interesting relation between $j_{\mathfrak{m}}(W)$ and $j_{\mathfrak{g}}(W)$, namely

$$j_{\mathfrak{g}}(W) = j_{\mathfrak{m}}^2(W)/\Delta_M(\exp W). \tag{4.7}$$

This turns out to play a key role in the proof of Theorem 4.9. Using the fact that $\Delta_M(\exp W) = \det \operatorname{Ad}_M(\exp -W) = \det e^{-\operatorname{ad}_{\mathfrak{m}} W}$ we obtain

$$\begin{aligned} j_{\mathfrak{m}}^2(W)/\Delta_M(\exp W) &= \det\left(\frac{\mathrm{id} - e^{-\operatorname{ad}_{\mathfrak{m}} W}}{\operatorname{ad}_{\mathfrak{m}} W}\right)^2 \det e^{\operatorname{ad}_{\mathfrak{m}} W} \\ &= \det\left(\frac{\sinh(\operatorname{ad}_{\mathfrak{m}} W/2)}{\operatorname{ad}_{\mathfrak{m}} W/2}\right)^2. \end{aligned}$$

The left side of (4.7) can be written in a similar fashion in terms of hyperbolic functions as

$$j_{\mathfrak{g}}(W) = \det\left(\frac{\mathrm{id} - e^{-\operatorname{ad}_{\mathfrak{g}} W}}{\operatorname{ad}_{\mathfrak{g}} W}\right) = \det\left(e^{-\operatorname{ad}_{\mathfrak{g}} W/2}\right) \det\left(\frac{\sinh(\operatorname{ad}_{\mathfrak{g}} W/2)}{\operatorname{ad}_{\mathfrak{g}} W/2}\right).$$

Since $\mathrm{SL}(2,\mathbb{R})$ is unimodular, $\det e^{-\operatorname{ad}_{\mathfrak{g}} W/2} = 1$, and, therefore, we get the following neat reformulation of (4.7):

$$\det\left(\frac{\sinh(\operatorname{ad}_{\mathfrak{g}} W/2)}{\operatorname{ad}_{\mathfrak{g}} W/2}\right) = \det\left(\frac{\sinh(\operatorname{ad}_{\mathfrak{m}} W/2)}{\operatorname{ad}_{\mathfrak{m}} W/2}\right)^2, \quad W \in \mathfrak{m}. \tag{4.8}$$

This condition seems to be worth studying in light of its critical role in the proof of the main result of this section. The domain of validity of Equation (4.8) is unknown to this author.

Before we embark on proving positivity of Kirillov's character for the orbit $\mathcal{O} = \mathrm{Ad}^*(\mathrm{SL}(2,\mathbb{R}))H^*$, let us mention that thanks to the existence of a real polarization, as in the case of nilpotent Lie groups, the coadjoint orbit $\mathcal{O}$ exhibits some affine structure, so we can expect Fourier analysis techniques to be very useful. In particular, the next example shows that the conclusion of Lemma 3.4 is still valid in this case.

Example 4.7. Recall the notation in Example 4.4. In the $H^*X^*Y^*$-coordinate system the coadjoint orbit $\mathcal{O} = \mathrm{Ad}^*(\mathrm{SL}(2,\mathbb{R}))H^*$ is a hyperboloid of one sheet. For $a \in \mathbb{R}^+$ and $b \in \mathbb{R}$, let

$$m = \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \in M.$$

Then $m \cdot H^* = H^* - abX^* + abY^*$. Thus, in the $H^*X^*Y^*$-coordinate system, $\mathrm{Ad}^*(M)H^* = \{(1, -c, c) \mid c \in \mathbb{R}\}$, which represents a line in $\mathfrak{sl}(2,\mathbb{R})^*$. See Figure 1. In this way, symplectic geometry can be seen as contributing to our proof of the positivity of Kirillov's character.

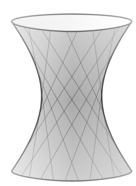

FIGURE 1. The coadjoint orbit $\mathcal{O} = \mathrm{Ad}^*(\mathrm{SL}(2,\mathbb{R}))H^*$ as a ruled surface

The coadjoint orbit $\mathcal{O}$ above corresponds to a principal series representation of $\mathrm{SL}(2,\mathbb{R})$ which is known to be tempered. So by Rossmann's theorem [20], Kirillov's character $\chi_{\mathcal{O}}$ is positive. To give a direct proof of this fact first we establish an analogue of Theorem 3.7 for $\mathrm{SL}(2,\mathbb{R})$.

Theorem 4.8. *Let U be a sufficiently small neighborhood of $\mathbf{0} \in \mathfrak{sl}(2,\mathbb{R})$ such that the restriction of $\exp\colon \mathfrak{sl}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R})$ to U is a diffeomorphism onto $\exp(U)$. Define $S = \{f \in C_c^\infty(\mathrm{SL}(2,\mathbb{R})) \mid \mathrm{supp}(f) \subset \exp(U)\}$. Then for any $f \in S$,*

$$\chi_{\mathcal{O}}(f) = \int_{G/M}\int_M f(xpx^{-1}) e^{i\ell(\log p)} \frac{\Delta_M^{-1/2}(p)}{\rho_{(G,M)}(x)}\, dp\, dxM \tag{4.9}$$

where $\chi_{\mathcal{O}}$ is the Kirillov character for the coadjoint orbit $\mathcal{O}$ through H^*.

Proof. To simplify the notation we shall write G and $\mathfrak{g}$ for $\mathrm{SL}(2,\mathbb{R})$ and $\mathfrak{sl}(2,\mathbb{R})$, respectively, and we let $\ell = H^*$. Assume $f \in S$. By the diffeomorphism $\mathcal{O} \cong G/R_\ell$, which carries one G-invariant measure to another, we make a change of variables to obtain

$$\begin{aligned}\chi_{\mathcal{O}}(f) &= \int_{\mathcal{O}} \int_{\mathfrak{g}} j_{\mathfrak{g}}^{1/2}(X) f(\exp X) e^{i\phi(X)}\, dX\, d\mu_{\mathcal{O}}(\phi) \\ &= \int_{G/R_\ell} \int_{\mathfrak{g}} j_{\mathfrak{g}}^{1/2}(X) f(\exp X) e^{ig\cdot\ell(X)}\, dX\, dgR_\ell.\end{aligned}$$

Recall from Example 4.4 that $\rho_{(G,M)}(xm)\rho_{(M,R_\ell)}(m) = \rho_{(G,M)}(x)$ for $x \in G$ and $m \in M$. Thus, by Corollary 4.5 and Theorem 3.5 we find

$$\begin{aligned}\chi_{\mathcal{O}}(f) &= \int_{G/M} \int_{M/R_\ell} \int_{\mathfrak{g}} j_{\mathfrak{g}}^{1/2}(X) f(\exp X) e^{ixm\cdot\ell(X)} \frac{1}{\rho_{(G,M)}(x)}\, dX\, dmR_\ell\, dxM \\ &= \Bigg(\int_{G/M} \int_{M/R_\ell} \int_{\mathfrak{g}/\mathfrak{m}} \int_{\mathfrak{m}} j_{\mathfrak{g}}^{1/2}(P+Y) f(\exp x\cdot(P+Y)) e^{im\cdot\ell(P+Y)} \\ &\qquad \frac{1}{\rho_{(G,M)}(x)}\, dP\, dY\, dmR_\ell\, dxM \Bigg).\end{aligned}$$

Here we have made the change of variable $x \mapsto x \cdot X$, and have used the invariance of the Lebesgue measure and $j_{\mathfrak{g}}$ under the adjoint action of G to simplify. Define $F_f^x(Y) = \int_{\mathfrak{m}} j_{\mathfrak{g}}^{1/2}(P+Y) f(\exp x\cdot(P+Y)) e^{i\ell(P+Y)} \frac{1}{\rho_{(G,M)}(x)}\, dP$ so that

$$\begin{aligned}\chi_{\mathcal{O}}(f) &= \int_{G/M} \int_{M/R_\ell} \int_{\mathfrak{g}/\mathfrak{m}} F_f^x(Y) e^{im\cdot\ell(Y) - i\ell(Y)}\, dY\, dmR_\ell\, dxM \\ &= \int_{G/M} \int_{(\mathfrak{g}/\mathfrak{m})^*} \int_{\mathfrak{g}/\mathfrak{m}} F_f^x(Y) e^{i\gamma(Y)}\, dY\, d\gamma\, dxM \\ &= \int_{G/M} F_f^x(\mathbf{0})\, dxM \\ &= \int_{G/M} \int_{\mathfrak{m}} j_{\mathfrak{g}}^{1/2}(P) f(\exp(x\cdot P)) e^{i\ell(P)} \frac{1}{\rho_{(G,M)}(x)}\, dP\, dxM,\end{aligned}$$

where in the second equality we have used the diffeomorphism $M/R_\ell \cong (\mathfrak{g}/\mathfrak{m})^*$ whose existence was proved in Lemma 3.4 and Example 4.7. The third equality is due to the Fourier inversion theorem. To proceed with the calculations, we observe that since $f \in S$ the change of variables formula applied to the restriction of exp to U implies

$$\int_G \frac{f(g)}{j_{\mathfrak{g}}(\log g)}\, dg = \int_U f(\exp X)\, dX.$$

Therefore,

$$\begin{aligned}\chi_{\mathcal{O}}(f) &= \int_{G/M} \int_{\mathfrak{m}} j_{\mathfrak{g}}^{1/2}(P) f(\exp(x\cdot P)) e^{i\ell(P)} \frac{1}{\rho_{(G,M)}(x)}\, dP\, dxM \\ &= \int_{G/M} \int_{M} j_{\mathfrak{g}}^{1/2}(\log p) \frac{f(xpx^{-1})}{j_{\mathfrak{m}}(\log p)} e^{i\ell(\log p)} \frac{1}{\rho_{(G,M)}(x)}\, dp\, dxM.\end{aligned}$$

The relation between the j functions in Equation 4.7 allows us to reduce Kirillov's character formula to

$$\chi_{\mathcal{O}}(f) = \int_{G/M} \int_M f(xpx^{-1}) e^{i\ell(\log p)} \frac{\Delta_M^{-1/2}(p)}{\rho_{(G,M)}(x)} \, dp \, dxM. \qquad \square$$

Finally, we are ready to prove our main result of this section.

Theorem 4.9. *Let the notation be as in Theorem 4.8. Then Kirillov's character $\chi_{\mathcal{O}}$ is positive on S in the sense that $\chi_{\mathcal{O}}(f * f^*) \geq 0$ whenever $f * f^* \in S$.*

Proof. Let $f * f^* \in S$. Then by (4.9) and writing out the definition of $f * f^*$ we have

$$\begin{aligned}\chi_{\mathcal{O}}(f * f^*) &= \int_{G/M} \int_M f * f^*(xpx^{-1}) e^{i\ell(\log p)} \frac{\Delta_M^{-1/2}(p)}{\rho_{(G,M)}(x)} \, dp \, dxM \\ &= \int_{G/M} \left(\int_M \int_G f(xpx^{-1}g) f^*(g^{-1}) e^{i\ell(\log p)} \frac{\Delta_M^{-1/2}(p)}{\rho_{(G,M)}(x)} \, dg \, dp \right) dxM \\ &= \int_{G/M} \left(\int_M \int_G f(xpx^{-1}g) \overline{f(g)} e^{i\ell(\log p)} \frac{\Delta_M^{-1/2}(p)}{\rho_{(G,M)}(x)} \, dg \, dp \right) dxM.\end{aligned}$$

Applying the change of variable $g \mapsto xg^{-1}$ to the innermost integral and using the unimodularity of G the last integral simplifies to

$$\int_{G/M} \left(\int_M \int_G f(xpg^{-1}) \overline{f(xg^{-1})} e^{i\ell(\log p)} \frac{\Delta_M^{-1/2}(p)}{\rho_{(G,M)}(x)} \, dg \, dp \right) dxM.$$

Now, we decompose the measure of G over M and G/M by combining Equation 4.3 with the quotient integral formula in Theorem 4.2.

$$\begin{aligned}\chi_{\mathcal{O}}(f * f^*) = \int_{G/M} \Bigg(\int_M \int_{G/M} \int_M & f(xpq^{-1}y^{-1}) \overline{f(xq^{-1}y^{-1})} e^{i\ell(\log p)} \\ & \frac{\Delta_M^{-1/2}(p)}{\rho_{(G,M)}(x)} \frac{\Delta_M^{-1}(q)}{\rho_{(G,M)}(y)} \, dq \, dyM \, dp \Bigg) dxM.\end{aligned}$$

Finally, we change the order of the two integrations in the middle and use the formulas in Definition 2.1 to make the change of variable $p \mapsto p^{-1}q$.

$$\begin{aligned}\chi_{\mathcal{O}}(f * f^*) &= \Bigg(\int_{G/M} \int_{G/M} \int_M \int_M f(xp^{-1}y^{-1}) \overline{f(xq^{-1}y^{-1})} e^{i\ell\left(\log p^{-1}q\right)} \\ &\qquad \frac{\Delta_M^{-1/2}(p)}{\rho_{(G,M)}(x)} \frac{\Delta_M^{-1/2}(q)}{\rho_{(G,M)}(y)} \, dq \, dp \, dyM \, dxM \Bigg) \\ &= \Bigg(\int_{G/M} \int_{G/M} \left| \int_M f(xp^{-1}y^{-1}) e^{-i\ell(\log p)} \Delta_M^{-1/2}(p) \, dp \right|^2 \\ &\qquad \frac{1}{\rho_{(G,M)}(x)} \frac{1}{\rho_{(G,M)}(y)} dyM \, dxM \Bigg) \geq 0. \qquad \square\end{aligned}$$

Example 4.10. Let E_{ij} be the 3×3 matrix with a one in the ij entry and zeros elsewhere. To simplify the notation, let us write F and G for $E_{11} - E_{22}$ and

$E_{11} - E_{33}$, respectively. Then a basis for the eight-dimensional Lie algebra $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{R})$ is given by

$$\mathfrak{B} = \{F, G, E_{12}, E_{23}, E_{13}, E_{21}, E_{32}, E_{31}\}.$$

Consider the polarizing subalgebra

$$\mathfrak{m} = \mathrm{Span}\{F, G, E_{12}, E_{23}, E_{13}\}$$

subordinate to F^* consisting of upper triangular matrices in $\mathfrak{sl}(3, \mathbb{R})$. For $W = fF + gG + e_3E_{12} + e_2E_{13} + e_1E_{23}$ in $\mathfrak{m}$, the matrix of $\mathrm{ad}_{\mathfrak{m}} W \colon \mathfrak{m} \to \mathfrak{m}$ with respect to the basis $\mathfrak{B}' = \{F, G, E_{12}, E_{23}, E_{13}\}$ of $\mathfrak{m}$ is

$$[\mathrm{ad}_{\mathfrak{m}} W]_{\mathfrak{B}'} = \left[\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline -2e_3 & -e_3 & 2f+g & 0 & 0 \\ e_1 & -e_1 & 0 & g-f & 0 \\ -e_2 & -2e_2 & -e_1 & e_3 & f+2g \end{array}\right].$$

Extend the basis $\mathfrak{B}'$ of $\mathfrak{m}$ to the basis $\mathfrak{B}$ of $\mathfrak{g}$. Then the matrix of $\mathrm{ad}_{\mathfrak{g}} W \colon \mathfrak{g} \to \mathfrak{g}$ with respect to the basis $\mathfrak{B}$ is

$$[\mathrm{ad}_{\mathfrak{g}} W]_{\mathfrak{B}} = \left[\begin{array}{c|ccc} & e_3 & 0 & -e_1 \\ & 0 & e_2 & e_1 \\ [\mathrm{ad}_{\mathfrak{m}} W]_{\mathfrak{B}'} & 0 & 0 & e_2 \\ & -e_2 & 0 & 0 \\ & 0 & 0 & 0 \\ \hline & -2f-g & 0 & e_1 \\ 0 & 0 & f-g & -e_3 \\ & 0 & 0 & -f-2g \end{array}\right].$$

As in the case of $\mathrm{SL}(2, \mathbb{R})$, this calculation shows that Equation (4.7) holds. Thus, by suitable adjustments to the proof of Theorem 4.9, one obtains the positivity of Kirillov's character for $\mathcal{O} = \mathrm{Ad}^*(\mathrm{SL}(3, \mathbb{R}))F^*$.

5. Construction of Representations for Nilpotent Lie Groups

In this section we exploit the positivity of Kirillov's character for a connected, simply connected nilpotent Lie group G to construct some representations of $G \times G$. Our method is analogous to the GNS construction in C*-algebra theory.

Let G be a connected, simply connected nilpotent Lie group. Then for any coadjoint orbit $\mathcal{O}$ of G, Kirillov's character $\chi_{\mathcal{O}} \colon C_c^\infty(G) \to \mathbb{C}$ is a positive distribution as we proved directly in Theorem 3.8. Moreover, it is straightforward to check that $\chi_{\mathcal{O}}(f_1 * f_2^*) = \overline{\chi_{\mathcal{O}}(f_2 * f_1^*)}$. Hence,

$$\langle f_1, f_2 \rangle_\chi = \chi_{\mathcal{O}}(f_1 * f_2^*) \tag{5.1}$$

defines a sesquilinear form on $C_c^\infty(G)$ that satisfies all the axioms for an inner product except for definiteness, that is, $\langle f, f \rangle_\chi = 0$ need not imply $f = 0$. The remaining axioms are enough to prove the Cauchy–Schwarz inequality

$$|\langle f_1, f_2 \rangle_\chi|^2 \leq \langle f_1, f_1 \rangle_\chi \langle f_2, f_2 \rangle_\chi, \tag{5.2}$$

from which it follows that the set $N = \{f \in C_c^\infty(G) \mid \langle f, f \rangle_\chi = 0\}$ is a vector subspace of $C_c^\infty(G)$. The formula

$$\langle f_1 + N, f_2 + N \rangle = \langle f_1, f_2 \rangle_\chi \tag{5.3}$$

defines an inner product on the quotient space $C_c^\infty(G)/N$. We let $\mathcal{H}_\sigma$ denote the Hilbert space completion of $C_c^\infty(G)/N$, and realize $\mathcal{H}_\sigma$ as a representation of $G\times G$ as follows. For $g_1, g_2 \in G$ and $f \in C_c^\infty(G)$ define $\lambda_{g_1} \circ \rho_{g_2}(f)$ by

$$\lambda_{g_1} \circ \rho_{g_2}(f)(x) = f(g_1^{-1} x g_2), \quad x \in G.$$

Now, we show that this extends to a unitary representation σ of $G \times G$ on $\mathcal{H}_\sigma$.

Lemma 5.1. *The left translation $(\lambda, C_c^\infty(G))$ and the right translation $(\rho, C_c^\infty(G))$ preserve the sesquilinear form* (5.1)*. That is, for any $g \in G$,*

$$\begin{aligned}\langle \rho_g f_1, \rho_g f_2\rangle_\chi &= \langle f_1, f_2\rangle_\chi, \text{ and,}\\ \langle \lambda_g f_1, \lambda_g f_2\rangle_\chi &= \langle f_1, f_2\rangle_\chi.\end{aligned}$$

Proof. The left translation invariance of the Haar measure of G gives $f_1 * f_2^*(x) = \int_G f_1(y)\overline{f_2(x^{-1}y)}\, dy = \int_G f_1(xy)\overline{f_2(y)}\, dy$. This, combined with the right translation invariance of the Haar measure, implies the right translation invariance of the sesquilinear form. To prove the second assertion, we observe that

$$\begin{aligned}\lambda_g f_1 * (\lambda_g f_2)^*(x) &= \int_G f_1(g^{-1}xy)\overline{f_2(g^{-1}y)}\, dy\\ &= \int_G f_1(g^{-1}xgy)\overline{f_2(y)}\, dy,\end{aligned}$$

and that the left translation invariance of the sesquilinear form, namely,

$$\langle \lambda_g f_1, \lambda_g f_2\rangle_\chi = \langle f_1, f_2\rangle_\chi$$

follows from the conjugation invariance of the character formula. In more details,

$$\begin{aligned}\chi_{\mathcal{O}}(c_g \cdot f) &= \int_{\mathcal{O}}\int_{\mathfrak{g}} f(g^{-1}\exp X g) e^{i\ell(X)}\, dX\, d\mu_{\mathcal{O}}(\ell)\\ &= \int_{\mathcal{O}}\int_G f(g^{-1}xg) e^{i\ell(\log x)}\, dx\, d\mu_{\mathcal{O}}(\ell)\\ &= \int_{\mathcal{O}}\int_G f(x) e^{ig^{-1}\cdot\ell(\log x)}\, dx\, d\mu_{\mathcal{O}}(\ell)\\ &= \int_{\mathcal{O}}\int_G f(x) e^{i\ell(\log x)}\, dx\, d\mu_{\mathcal{O}}(\ell) = \chi_{\mathcal{O}}(f),\end{aligned}$$

where we have used the G-invariance of the canonical measure $\mu_{\mathcal{O}}$ in the second to last equality. $\square$

Corollary 5.2. *The translation maps $(\lambda, C_c^\infty(G))$ and $(\rho, C_c^\infty(G))$ extend to the unitary representations $(L, \mathcal{H}_\sigma)$ and $(R, \mathcal{H}_\sigma)$ of G via*

$$L_g[f] = [\lambda_g f] \text{ and } R_g[f] = [\rho_g f].$$

Therefore, $(\sigma, \mathcal{H}_\sigma)$ defined by $\sigma(g_1, g_2) = L_{g_1} \circ R_{g_2}$ is a unitary representation of $G \times G$.

Proof. The well definedness and unitarity of the operators L_g and R_g are immediate from the Cauchy–Schwarz inequality (5.2) and Lemma 5.1. Moreover, the strong continuity of the left and right regular representations $(\lambda, C_c^\infty(G))$ and $(\rho, C_c^\infty(G))$ at $g = e_G$ is transferred to $(L, \mathcal{H}_\sigma)$ and $(R, \mathcal{H}_\sigma)$ via the inner product formula (5.3) and this completes the proof. $\square$

Suppose that the coadjoint orbit $\mathcal{O}$ corresponds, under the Kirillov quantization, to (the class of) the irreducible unitary representation $(\pi, \mathcal{K}_\pi)$ of G. A natural question that arises is how the Kirillov G-representation associated to $\mathcal{O}$, namely $\mathcal{K}_\pi$, and our $G\times G$-representation $\mathcal{H}_\sigma$—obtained by applying the GNS construction to the positive distribution $\chi_{\mathcal{O}}$—are related. The answer is given by the next theorem.

Theorem 5.3. *Let* $\mathrm{HS}(\mathcal{K}_\pi)$ *denote the space of Hilbert–Schmidt operators on* $\mathcal{K}_\pi$*, and let* $(\eta, \mathrm{HS}(\mathcal{K}_\pi))$ *be the representation of* $G\times G$ *given by* $\eta(x,y)(T) = \pi(x)T\pi(y^{-1})$*. We have the following* $G \times G$*-equivariant isometric isomorphisms*

$$\mathcal{H}_\sigma \cong \mathrm{HS}(\mathcal{K}_\pi) \cong \mathcal{K}_\pi \otimes \mathcal{K}_\pi^*.$$

Proof. First, recall from Theorem 2.6 the fact that $\chi_{\mathcal{O}}(f) = \mathrm{Tr}\,\pi(f)$. Using this, we have $\langle f, f\rangle_\chi = \|\pi(f)\|^2_{\mathrm{HS}}$. Hence, $N = \{f \in C_c^\infty(G) \mid \langle f, f\rangle_\chi = 0\} = \{f \in C_c^\infty(G) \mid \pi(f) = 0\}$. Thus, $[f] \mapsto \pi(f)$ gives a well-defined, injective, linear map $C_c^\infty(G)/N \to \mathrm{HS}(\mathcal{K}_\pi)$. Since

$$\begin{aligned}\pi(\sigma(g_1, g_2)f) &= \int_G f(g^{-1}xg_2)\pi(x)\,dx = \int_G f(x)\pi(g_1xg_2^{-1})\,dx \\ &= \pi(g_1)\pi(f)\pi(g_2^{-1}) = \eta(g_1, g_2)\pi(f),\end{aligned}$$

$[f] \mapsto \pi(f)$ extends to a $G \times G$-equivariant isometric isomorphism of $\mathcal{H}_\sigma$ and $\mathrm{HS}(\mathcal{K}_\pi)$. (For surjectivity, we refer to [5, Section 18.8]). This establishes the first isomorphism. The second isomorphism, between $\mathrm{HS}(\mathcal{K}_\pi)$ and $\mathcal{K}_\pi \otimes \mathcal{K}_\pi^*$, is given by the very definition of the tensor product. □

Finally, we make a remark about applying the constructions in this section to non-nilpotent Lie groups. Note that even when the Kirillov character formula is positive, it does not obviously determine a representation. This is because the exponential map is neither injective nor surjective in general, and the test functions supported in a fixed neighborhood of the identity element of the group do not necessarily form an algebra under the convolution operation. This leads to the following question by Higson [10] "Is there a useful concept of partial representation corresponding to the partially-defined Kirillov character?"

6. Conclusion

By a direct proof, we have shown that Kirillov's character defines a positive trace on the convolution algebra of smooth, compactly supported functions on a connected, simply connected nilpotent Lie group G. Then using this and the GNS construction we have produced certain unitary representations of $G \times G$. We have also shown how the same methods apply to coadjoint orbits of other Lie groups, including $\mathrm{SL}(2,\mathbb{R})$, under a few additional conditions that are automatically satisfied in the nilpotent case. One hypothesis is the geometric condition required by Lemma 3.4 (which is indeed a statement about Lagrangian fibrations); the other is the existence of real polarizations satisfying Equation (4.8).

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